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Abstract

One of the most fruitful and elegant approach (known as Kolosov-Muskhelishvili formulas) for plane isotropic elastic problems is to use two complex-valued holomorphic potentials. In this paper, the algebra of real quaternions is used in order to propose in three dimensions, an extension of the classical Muskhelishvili formulas. The starting point is the classical harmonic potential representation due to Papkovich and Neuber. Alike the classical complex formulation, two monogenic functions very similar to holomorphic functions in 2D and conserving many of interesting properties, are used in this contribution. The completeness of the potential formulation is demonstrated rigorously. Moreover, body forces, residual stress and thermal strain are taken into account as a left side term. The obtained monogenic representation is compact and a straightforward calculation shows that classical complex representation for plane problems is embedded in the presented extended formulas. Finally the classical uniqueness problem of the Papkovich-Neuber solutions is overcome for polynomial solutions by fixing explicitly linear dependencies.

Keywords: Isotropic elasticty, Quaternions, Monogenic potential, Meshless, Residual stress, Thermal load

Table 1: Nomenclature

Linear elasticity	
x_1, x_2, x_3	Cartesian coordinates
$\underline{\boldsymbol{\sigma}}$	Stress tensor
$\underline{\boldsymbol{\epsilon}}$	Total strain tensor
$\underline{\boldsymbol{\epsilon}}^{th}$	Thermal strain tensor
$\underline{\boldsymbol{\epsilon}}^{res}$	Residual strain tensor
$\underline{\boldsymbol{\epsilon}}^{*}$	Strain tensor $\underline{\boldsymbol{\epsilon}}^{*} = \underline{\boldsymbol{\epsilon}}^{th} + \underline{\boldsymbol{\epsilon}}^{res}$
$\underline{\boldsymbol{\sigma}}^{*}$	Auxiliary stress tensor $\underline{\boldsymbol{\sigma}}^{*} = \underline{\boldsymbol{\sigma}} + \lambda \mathrm{tr}\left(\underline{\boldsymbol{\epsilon}}^{*}\right) \underline{\boldsymbol{I}} + 2\mu \underline{\boldsymbol{\epsilon}}^{*}$
$\boldsymbol{u}$	Displacement field
$\boldsymbol{f}_b$	Body forces
Ω	Elastic medium (open subset of $\mathbb{R}^3$)
$\partial\Omega$	Boundary of Ω
$\partial\Omega_u$	Subpart of $\partial\Omega$ where displacement is imposed

$\partial\Omega_\sigma$	Subpart of $\partial\Omega$ where surface traction is imposed
$\boldsymbol{n}$	Normal vector
$\boldsymbol{u}_b$	Displacement imposed on $\partial\Omega_u$
$\boldsymbol{T}_b$	Surface traction imposed on $\partial\Omega_\sigma$
T_0	Temperature of the released configuration
T	Temperature of the body
λ, μ	Lamé's coefficients
E, ν	Young modulus and Poisson coefficient
Potential theory	
$\boldsymbol{\Gamma}$	Vector potential (left side term of the Lamé-Navier equation)
$\boldsymbol{\Gamma}^*$	Vector potential related to $\boldsymbol{\Gamma}$
$\boldsymbol{F}$	Galerkin vector potential
$\boldsymbol{f}, G, h$	Papkovich-Neuber potentials
Φ	Monogenic potential
$\Theta, \widehat{\Psi}$	Anti-monogenic potentials
Λ	Monogenic constant
A_n^l	Monogenic polynomials of degree n ($n \in \mathbb{N}, l = 0, .., n$)

1. Introduction

1.1. Applications of potential theory

Well known numerical methods such as Finite Element Method (FEM) or Boundary Element Method (BEM) enable to solve various complex mechanical problems including non-linear problems (plasticity or other non-linear behaviors, contact problems, large displacements etc.). Isotropic linear elasticity is nevertheless a frequent problem in mechanical engineering. Potential theory developed since the late 19^{th} century is still widely used in linear elasticity in 2D and 3D. Barber (2003) presents an overview of the fundamental potential theory for elasticity related among others to Airy, Boussinesq, Green, Zerna, Galerkin, Papkovich and Neuber names. New potential formulations for instance developed by Kashtalyan and Rushchitsky (2009) deal with inhomogenous media.

Many practical applications rely on potential theory. Stress Intensity Factors (SIF) in the framework of linear fracture mechanics have been intensively studied. For example Sneddon and Lowengrub (1969) or Kassir and Sih (1973, 1975) proposed various analytical solutions based on potential theory. Dual integral equations were intensively used for mixed boundary value problems that arise in potential theory adapted for crack problems. An overview of useful methods is given by Sneddon (1966). Fully analytical or semi-analytical solutions have also been established for various elastic problems using potential theory. For instance, Ying et al. (1996) applied potential theory for a pressure vessels and piping. Chau and Wei (2000) proposed a semi-analytical solution (relying on truncated expansions into series of the potentials) of a finite solid circular cylinder subjected to arbitrary surface load. More recently potential theory has been used for applied industrial investigations. In the field of rolling process for instance, coupled thermo-elastic inverse solutions that interpret (in real time) measurements of stress and temperature done under the surface of a cylindrical tool have been proposed in 2D by Weisz-Patrault et al. (2011, 2012a, 2013a) and in 3D by Weisz-Patrault et al. (2013b, 2014). Thus, the contact between the product and the tool can be characterized during the process. Experimental tests that confirm the feasibility of such an approach have been performed by Weisz-Patrault et al. (2012b) and Legrand et al. (2012, 2013). This kind of recent works contributes to renew the interest for potential theory because of their practical and technical content.

Furthermore, numerical methods can also be developed on the basis of potential theory. Hintermüller et al. (2009) proposed a 3D potential based numerical method for cracks and contact problems. Potential theory adapted for numerical methods are completely meshless and can be suitable for problems where very steep stress gradients are obtained avoiding mesh refinement and long computation times issues that arise with FEM for instance. Cruse (1969) proposed such a numerical algorithm based on potentials and singular integral equations. Morales et al. (2013) proposed more recently a potential based numerical solution for 2D problems, and Morales et al. (2012) focuses on numerical uniqueness of the Boussinesq and Timpe solutions.

1.2. Motivations for extended Muskhelishvili formulas

For plane problems one of the most elegant and fruitful approach has been developed by Muskhelishvili (1953b). Complex plane is used and holomorphic $\mathbb{C}$-valued potentials are derived from bi-harmonic Airy potential and Goursat theorem. A presentation of the theory and practical methods has been given by Lu (1995). The main advantages are related to the holomorphy of the involved potentials, indeed expansion into series, Cauchy formula and conformal mapping techniques are available as well as singular integral equation techniques studied by Muskhelishvili (1953a). Usually, for three-dimensional problems $\mathbb{R}$-valued harmonic or bi-harmonic potentials are used, known as Galerkin vector potential and Papkovich-Neuber potentials initially introduced by Papkovich (1932) and re-discovered by Neuber (1934). These potential representations are complete, thus one can prove the existence of the potentials as studied by Mindlin (1936); Gurtin (1962); Stippes (1969); Cong and Steven (1979a); Millar (1984); Hackl and Zastrow (1988). Complete general solutions are also studied in the fundamental works by Slobodyansky (1954, 1959) and Wang et al. (2008) among others.

On the basis of Papkovich-Neuber potentials, this paper aims at establishing a generalized Muskhelishvili formula in three dimensions. There is no direct extension of the complex plane in 3D. However, the four dimensional algebra of quaternions (Definition 1) is a convenient extension of the complex plane. Extensive work has been done in this field and a suitable extension in higher dimensions of holomorphic functions has been defined and studied intensively. For instance the book of Gürlebeck et al. (2007) gathers standard knowledge about the algebra of real quaternions. A class of functions, called monogenic (Definition 3), presents interesting similarities with holomorphic functions defined in the complex plane. Thus several advantages of the classical formulas of Muskhelishvili (1953b) in 2D are transposed in 3D with the presented potential formulation. Indeed, monogenic power series expansions studied for instance by Malonek (1990); Bock and Gürlebeck (2010); Bock (2012b) and Laurent series expansions (see e.g. van Lancker (1999); Bock (2012a)) as well as the Cauchy formula (e.g. Brackx et al.

(1982)) are still available. Conformal mapping technics are more limited than in 2D, but Möbius transformations are still available as detailed by Sudbery (1979).

A second motivation is the disadvantage of Papkovich-Neuber representation that arises if polynomial solutions of exact degree n are considered for the displacement field. Indeed, Bauch (1981) showed that if very classical spherical harmonics are used for the Papkovich-Neuber potentials then $8n + 4$ polynomial solutions are generated, but the dimension of the subspace of polynomial solutions of degree n is only $6n + 3$. Thus, many solutions obtained with Papkovich-Neuber representation are linear dependent which can cause numerical stability problems. But fixing these dependencies in explicit formulas is very difficult. However, Bock and Gürlebeck (2009b) already proposed a representation of displacement field by means of two monogenic functions which is similar to the representation demonstrated in this paper. Then Bock and Gürlebeck (2009a) demonstrated that $8n + 8$ polynomial solutions are generated by considering spherical monogenics for the two monogenic functions. But $2n + 5$ are linear dependent and explicit formulas have been given. Thus, monogenic representations present the significant advantage (compared with classical Papkovich-Neuber representation) of allowing explicit formulas of linear dependencies when spherical harmonics (or monogenics) are used for the potentials. Thus, numerical stability is expected to be much better for numerical applications.

In this paper, the existence of the two monogenic potentials is proven a priori by using only mathematical tools related to differentials calculus alike classical proofs of Airy potentials, Muskhelishvili formulas or Papkovich-Neuber representation. Thus completeness is demonstrated and an elegant and very compact representation of the displacement and stress fields is obtained. Moreover body forces, thermal strain and residual stress are taken into account in the potential representation. Finally in section 6, polynomial solutions are constructed and it is shown how the redundancy of polynomial systems can be overcome.

Furthermore Piltner (1987, 1988, 1989) contributed significantly to potential theory by developing an alternative complete representations of 3D isotropic elasticity based on complex functions. Piltner (2001) provided an overview of complex methods. He was using six holomorphic functions depending on three complex variables, defined as complex-valued linear functions on $\mathbb{R}^3$. These representations cover under certain restrictions on the parameters the known representation formulas for the plane case and there are also results to restrict the number of complex variables to one. Without going too much into the details it should be mentioned that these representations are deeply related to each other. The linear functions used by Piltner can be found in Whittaker (1903) and in the book by Whittaker and Watson (1927) as a tool to describe spherical harmonics. In this way they are related also to the representation of Legendre polynomials and associate Legendre functions which are nowadays mainly used for this purpose (see for instance Sansone (1959)).

In this paper, a different framework is used (algebra of real quaternions instead of complex plane) regarding to the advantages listed in this section. It should be noted that another potential solution for 3D Neumann and Dirichlet problems (surface tractions or displacements imposed at the surface) for a general elastic body is described in the book of Bui (2006). The solution relies on the Kelvin-Somigliana or Kupradze-Bashelishvili tensors (equivalent to the Green tensor for elastostatic) introduced by Kupradze (1965). On this basis a simple or double layer potential vector and an integral equation has been solved analytically (in the form of an absolutely convergent series) by Pham (1967). In this paper the extended Muskhelishvili formulas are not derived from these potentials, because this method does not rely on harmonic analysis.

1.3. Geometrical restrictions

Complete representations for displacements require geometrical restrictions due to constructions. These restrictions are relatively weak and related to the boundary value problem that has to be solved. More serious is the problem of redundancy in the representation formulae because this avoids the uniqueness of the representations. Analyzing for instance the classical Papkovich-Neuber representation then it is known already for a long time that under certain additional assumptions only three of the four harmonic functions are needed. Sokolnikoff (1956) showed that one of the three harmonic functions in the vector potential can be omitted (set to be zero) if the domain is normal with respect to the corresponding direction. The scalar potential can be removed if for $\nu \neq \frac{1}{4}$ the domain is star-shaped. What is not so much discussed is the question whether additional assumptions are necessary if one of the four functions should only be expressed as a linear combination of the other three. A good survey on results about the uniqueness of the representations can be found also in Cong (1995).

This idea becomes more important when it is tried to construct better structured representation formulae. Taking the classical Kolosov-Muskhelishvili formulae as a starting point the improved structure is given by the formulation based on two holomorphic functions. This representation can be generalized to the three-dimensional case and was done in Bock and Gürlebeck (2009b,a) by using the theory of quaternion-valued holomorphic (monogenic) functions. In these papers it is the goal to find finally polynomial approximations for displacements and stresses, respectively. Collecting all geometrical restrictions final results are valid for star-shaped domains.

This paper aims at demonstrating generalized Kolosov-Mushelishvili formulae with thermal strain and residual stress, in a constructive way. For this reason, as explained below in detail the elastic domain is assumed to be normal with respect to the x_1-direction (Definition 4). The proof of completeness of the representation using two monogenic functions is related to Theorem 1, which is valid for domains normal with respect to the x_1-direction. Thus the representation demonstrated in this paper is proved to be complete on domains normal with respect to the x_1-direction. This constitutes a large class of shapes for the elastic body. The paper generalizes the applicability of the considered representations by adding domains normal with respect to the x_1-direction to the already available class of star-shaped domains. For domains that are not normal with respect to the x_1-direction, if the body can be split into subparts that meet the geometrical restrictions, one could solve the elastic problem on each subpart with a parametrized boundary condition at the junction of two successive parts, the final solution would be obtained by ensuring the continuity of displacements and the tensile vector at each interface.

However if monogenic potentials are well defined on the entire space $\mathbb{R}^3$ and not only on the studied domain Ω, then the representation is proven even if Ω does not fulfill the geometrical restrictions. This can be useful for practical applications, because most of the time spherical monogenics are used for the potentials (and are well defined in $\mathbb{R}^3$), therefore practically for many common cases there is no geometrical restrictions.

1.4. Notations and structure of the paper

Real vectors are classically written in bold. The quaternionic counter-parts (although representing the same vectors) are written with the same letter but not in bold alike classical notations for complex representation in 2D. Usually (x_0, x_1, x_2, x_3) denote the coordinates of points in the algebra of real quaternions, however in this paper (x_1, x_2, x_3, x_4) is used instead in order to be consistent with classical mechanical notations, in this way a point of the real 3D space is denoted by (x_1, x_2, x_3) and displacement, stress and strain tensors are indexed with $\{1, 2, 3\}$. Real second order tensors are underlined and bold. Notations are listed in Table 1. In this paper Ω denotes a connected subset of $\mathbb{R}^3$ representing the studied elastic body. In the whole paper Ω has a piecewise smooth boundary.

In Section 2, Papkovich-Neuber potentials are introduced with body forces, thermal strain and residual stress. Then in Section 3, the necessary mathematical results are stated and demonstrated. This latter section aims at establishing a rigorous framework for the monogenic potential representation. Thus, in Section 4, the extension of Muskhelishvili complex formulas is proved in 3D by demonstrating the existence of two monogenic potentials. In Section 5, the classical 2D complex equation set is derived from the 3D monogenic representation in order to show that the latter is a straightforward extension of the former. Finally, in Section 6 complete orthogonal systems of monogenic polynomials are used to construct a complete system of polynomial solutions to the Lamé-Navier equations. As usual there are some linearly dependent polynomials and it will be shown explicitly how the dependent polynomials can be removed from the system.

2. Classical complete representations

Let consider an elastic body represented by Ω (a connected subset of $\mathbb{R}^3$). Both thermal (superscript *th*) and residual (superscript *res*) strain tensors are considered, resulting in additional thermal and residual stresses. Displacements $\boldsymbol{u}_b$ and surface traction $\boldsymbol{T}_b$ are respectively prescribed on subparts of the boundary $\partial\Omega_u$ and $\partial\Omega_\sigma$ such as $\partial\Omega = \partial\Omega_u \cup \partial\Omega_\sigma$. Thus the isotropic elastic problem on Ω with body force $\boldsymbol{f}_b$ consists in solving the following equation set:

$$\left\{\begin{array}{ll} \mathbf{div}\left(\underline{\boldsymbol{\sigma}}\right) = -\boldsymbol{f}_b & \text{Equilibrium} \\ \underline{\boldsymbol{\sigma}} = \lambda \mathrm{tr}\left(\underline{\boldsymbol{\epsilon}}^e\right)\underline{\boldsymbol{I}} + 2\mu\underline{\boldsymbol{\epsilon}}^e & \text{Isotropic elastic behaviour} \\ \underline{\boldsymbol{\epsilon}} = \frac{1}{2}\left(\underline{\boldsymbol{\nabla}}(\boldsymbol{u}) + \underline{\boldsymbol{\nabla}}(\boldsymbol{u})^T\right) & \text{Compatibility} \\ \underline{\boldsymbol{\epsilon}}^{th} = \alpha(T - T_0)\underline{\boldsymbol{I}} & \text{Isotropic thermal behaviour} \\ \underline{\boldsymbol{\epsilon}}^e = \underline{\boldsymbol{\epsilon}} - \underline{\boldsymbol{\epsilon}}^{th} - \underline{\boldsymbol{\epsilon}}^{res} & \text{Elastic strain tensor} \\ (x_1, x_2, x_3) \in \partial\Omega_u,\ \boldsymbol{u}(x_1, x_2, x_3) = \boldsymbol{u}_b(x_1, x_2, x_3) & \text{Boundary conditions: displacements} \\ (x_1, x_2, x_3) \in \partial\Omega_\sigma,\ \underline{\boldsymbol{\sigma}}.\boldsymbol{n} = \boldsymbol{T}_b(x_1, x_2, x_3) & \text{Boundary conditions: surface traction} \end{array}\right. \tag{1}$$

It should be noted that body forces $\boldsymbol{f}_b$, temperature field T and residual strain $\underline{\boldsymbol{\epsilon}}^{res}$ are assumed to be known. The elastic calculation does not evaluate these latter quantities but use them as inputs alike loads. Displacement field of elastic problems on a domain Ω can be written by means of the classical vector potential $\boldsymbol{F}$ introduced by Galerkin (1930) and proven to be complete for instance by Westergaard (1952).

$$2\mu\boldsymbol{u} = 2(1 - \nu)\boldsymbol{\Delta}\,\boldsymbol{F} - \boldsymbol{\nabla}\mathrm{div}\,\boldsymbol{F} \tag{2}$$

A constitutive equation for the Galerkin vector is obtained by verifying the equilibrium equation. Thus, the Lamé-Navier equation (which is obtained by writing the equilibrium as a function of displacements) is used:

$$\boldsymbol{\Delta}\,\boldsymbol{u} + \frac{\lambda+\mu}{\mu}\boldsymbol{\nabla}\mathrm{div}\,\boldsymbol{u} = \alpha\left(\frac{3\lambda+2\mu}{\mu}\right)\boldsymbol{\nabla}T + \frac{\boldsymbol{E}^{res}}{\mu} - \frac{\boldsymbol{f}_b}{\mu} \tag{3}$$

Where:

$$\begin{aligned}\boldsymbol{E}^{res} \quad &= \left[(\lambda+2\mu)\frac{\partial\epsilon_{11}^{res}}{\partial x_1} + \lambda\left(\frac{\partial\epsilon_{22}^{res}}{\partial x_1} + \frac{\partial\epsilon_{33}^{res}}{\partial x_1}\right) + 2\mu\left(\frac{\partial\epsilon_{12}^{res}}{\partial x_2} + \frac{\partial\epsilon_{13}^{res}}{\partial x_3}\right)\right]\boldsymbol{e}_1 \\ &+ \left[(\lambda+2\mu)\frac{\partial\epsilon_{22}^{res}}{\partial x_2} + \lambda\left(\frac{\partial\epsilon_{11}^{res}}{\partial x_2} + \frac{\partial\epsilon_{33}^{res}}{\partial x_2}\right) + 2\mu\left(\frac{\partial\epsilon_{12}^{res}}{\partial x_1} + \frac{\partial\epsilon_{23}^{res}}{\partial x_3}\right)\right]\boldsymbol{e}_2 \\ &+ \left[(\lambda+2\mu)\frac{\partial\epsilon_{33}^{res}}{\partial x_3} + \lambda\left(\frac{\partial\epsilon_{11}^{res}}{\partial x_3} + \frac{\partial\epsilon_{22}^{res}}{\partial x_3}\right) + 2\mu\left(\frac{\partial\epsilon_{13}^{res}}{\partial x_1} + \frac{\partial\epsilon_{23}^{res}}{\partial x_2}\right)\right]\boldsymbol{e}_3\end{aligned} \tag{4}$$

There exists $\boldsymbol{\Gamma}$ such as:

$$\boldsymbol{\Delta}\,\boldsymbol{\Delta}\,\boldsymbol{\Gamma} = \alpha\left(\frac{3\lambda+2\mu}{\mu}\right)\boldsymbol{\nabla}T + \frac{\boldsymbol{E}^{res}}{\mu} - \frac{\boldsymbol{f}_b}{\mu}$$

Thus the classical constitutive equation for the Galerkin vector is obtained:

$$\boldsymbol{\Delta}\,\boldsymbol{\Delta}\,\left[\boldsymbol{F} - \boldsymbol{\Gamma}\right] = 0 \tag{5}$$

The main disadvantage of the Galerkin vector representation is that three scalar bi-harmonic functions are needed. On can simplify significantly this representation. Let introduce the harmonic vector $\boldsymbol{f}$:

$$\boldsymbol{f} = \frac{1}{2}\boldsymbol{\Delta}\,\left[\boldsymbol{F} - \boldsymbol{\Gamma}\right] \tag{6}$$

Let introduce $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$, thus $\boldsymbol{x}.\boldsymbol{f} = x_1 f_1 + x_2 f_2 + x_3 f_3$. A straightforward calculation gives (since $\boldsymbol{f}$ is harmonic):

$$\Delta\left(\boldsymbol{x}.\boldsymbol{f}\right) = 2\mathrm{div}\,\boldsymbol{f} = \mathrm{div}\,\boldsymbol{\Delta}\,\left[\boldsymbol{F} - \boldsymbol{\Gamma}\right] = \Delta\,\mathrm{div}\,\left[\boldsymbol{F} - \boldsymbol{\Gamma}\right] \tag{7}$$

Thus by integrating the Laplacian operator in (7) there exists a real harmonic function h such as:

$$G = \boldsymbol{x}.\boldsymbol{f} + h = \mathrm{div}\,\left[\boldsymbol{F} - \boldsymbol{\Gamma}\right] \tag{8}$$

It is easily verified from (7) that:

$$\Delta\Delta\,G = 2\mathrm{div}\,\boldsymbol{\Delta}\,\boldsymbol{f} = 0 \tag{9}$$

Hence from (2):

$$2\mu\boldsymbol{u} = 4(1-\nu)\boldsymbol{f} - \boldsymbol{\nabla}G + \boldsymbol{\Gamma}^* \tag{10}$$

Where:

$$\boldsymbol{\Gamma}^* = 2(1-\nu)\boldsymbol{\Delta}\,\boldsymbol{\Gamma} - \boldsymbol{\nabla}\mathrm{div}\,\boldsymbol{\Gamma}$$

Finally the complete Papkovich-Neuber representation is obtained:

$$\begin{cases} 2\mu u_1 = 4(1-\nu)f_1 - \dfrac{\partial G}{\partial x_1} + \Gamma_1^* \\ 2\mu u_2 = 4(1-\nu)f_2 - \dfrac{\partial G}{\partial x_2} + \Gamma_2^* \\ 2\mu u_3 = 4(1-\nu)f_3 - \dfrac{\partial G}{\partial x_3} + \Gamma_3^* \end{cases} \tag{11}$$

This potential representation is the basis of the extended Muskhelishvili formulas that are proven in this paper.

3. Mathematical results

This section presents the mathematical preliminaries for the quaternionic representation. Some definitions and classical theorems are reminded for sake of clarity. This section does not aim at presenting a mathematical discussion but presents only the useful results for establishing the three-dimensional extension of the classical complex formulas of Muskhelishvili (1953b).

Definition 1 (Algebra of real quaternions)**.** Let $\mathbb{H}$ denote the non-commutative algebra of real quaternions:

$$\mathbb{H} = \left\{x = x_1 + ix_2 + jx_3 + kx_4, (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\right\}$$

Where i, j and k are the imaginary numbers verifying following multiplication rules:

$$i^2 = j^2 = k^2 = -1 \;\Big|\; ij = -ji = k \;\Big|\; jk = -kj = i \;\Big|\; ki = -ik = j$$

Of course $\mathbb{H} \simeq \mathbb{R}^4$. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$ be an orthonormal basis of $\mathbb{R}^4$. For all $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 \in \mathbb{R}^4$, the corresponding quaternion is $x = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$. Furthermore, for all $x \in \mathbb{H}$ following quantities are classically defined:

(i) The scalar part of x is $\mathrm{Sc}\,[x] = x_1$

(ii) The vectorial part is $\mathrm{Vec}\,[x] = \underline{x} = ix_2 + jx_3 + kx_4$

(iii) The conjugate of x is $\overline{x} = x_1 - \underline{x} = x_1 - ix_2 - jx_3 - kx_4$

(iv) The k-involution of x is $\widehat{x} = -kxk = x_1 - ix_2 - jx_3 + kx_4$

(v) The norm is $|x| = \sqrt{x\overline{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$

(vi) The inverse of $x \neq 0$ is $x^{-1} = \overline{x}/|x|^2$

The reduced quaternion set denoted by $\mathcal{A} \simeq \mathbb{R}^3$ is defined as the subset of $\mathbb{H}$ generated by $(1, i, j)$:

$$\mathcal{A} = \left\{x = x_1 + ix_2 + jx_3, (x_1, x_2, x_3) \in \mathbb{R}^3\right\}$$

It should be noted that $\mathcal{A}$ is only a real vector space and not a sub-algebra of $\mathbb{H}$ because if x and y are two elements of $\mathcal{A}$ the product $xy \notin \mathcal{A}$ (of course $xy \in \mathbb{H}$). Moreover, let be $x = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$.

Let Ω be an open subset of $\mathbb{R}^3 \simeq \mathcal{A}$ with piecewise smooth boundary. An $\mathbb{H}$-valued function $v : \begin{cases} \Omega \to \mathbb{H} \\ x \mapsto v(x) \end{cases}$, is defined with four $\mathbb{R}$-valued functions $v_l : \begin{cases} \Omega \to \mathbb{R} \\ x \mapsto v_l(x) \end{cases}$ $(l \in \{1; ...; 4\})$, such as $v = v_1 + iv_2 + jv_3 + kv_4$. Continuity, differentiability or integrability of v are defined coordinate-wisely. All functions considered in the following will be taken either in the right $\mathbb{H}$-linear or in the right $\mathbb{R}$-linear Hilbert space of square-integrable $\mathbb{H}$-valued functions denoted by $L^2(\Omega, \mathbb{H})$ or $L^2(\Omega, \mathbb{R})$. For a detailed discussion of the function spaces and the corresponding inner product see e.g. Gürlebeck et al. (2007).

Definition 2. The generalized Cauchy-Riemann operator and its conjugate are defined by:

$$\begin{cases} \overline{\partial} = \dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial \underline{x}} = \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} + j\dfrac{\partial}{\partial x_3} \\ \partial = \dfrac{\partial}{\partial x_1} - \dfrac{\partial}{\partial \underline{x}} = \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2} - j\dfrac{\partial}{\partial x_3} \end{cases} \tag{12}$$

Definition 3 (Monogenic, Anti-monogenic, Monogenic constant)**.** A function $v \in C^1(\Omega, \mathbb{H})$ is called *monogenic* in $\Omega \subset \mathbb{R}^3$ if

$$\overline{\partial} v = 0 \text{ in } \Omega \text{ (or equivalently } v \in \ker \overline{\partial} \text{ in } \Omega). \tag{13}$$

Conversely, a function $v \in C^1(\Omega, \mathbb{H})$ is called *anti-monogenic* in $\Omega \subset \mathbb{R}^3$ if

$$\partial v = 0 \text{ in } \Omega \text{ (or equivalently } v \in \ker \partial \text{ in } \Omega). \tag{14}$$

Furthermore, a function $v \in C^1(\Omega, \mathbb{H})$ is called *monogenic constant* in $\Omega \subset \mathbb{R}^3$ if

$$\overline{\partial} v = \partial v = 0 \text{ in } \Omega \text{ (or equivalently } v \in \ker \overline{\partial} \cap \ker \partial \text{ in } \Omega). \tag{15}$$

Generalized Cauchy-Riemann operators are analogous to the well known Cauchy-Riemann operators in complex analysis, and monogenic (resp anti-monogenic) functions are analogous to holomorphic (resp anti-holomorphic) functions in 2D. A conversion of a given monogenic function into an anti-monogenic function and vice versa can be done via the following proposition.

Proposition 1. Let $v = v_1 + iv_2 + jv_3 + kv_4 \in C^1(\Omega, \mathbb{H})$ be a monogenic function in $\Omega \subset \mathbb{R}^3$. The function

$$\widehat{v} = v_1 - iv_2 - jv_3 + kv_4 \tag{16}$$

defines an anti-monogenic function in Ω (such that $\partial\widehat{v} = 0$). Conversely if v is anti-monogenic $\widehat{v}$ is monogenic (such that $\overline{\partial}\widehat{v} = 0$)

Proof. A straightforward calculation using the definition 3 gives:

$$\partial\widehat{v} = \widehat{\overline{\partial} v} \text{ and } \partial v = \overline{\partial}\widehat{v} \tag{17}$$

which demonstrates the proposition. □

This latter proposition enables to simplify significantly calculations in the following. Here, it should be emphasized that in the complex case the conjugation of an holomorphic function $v \in C^1(\Omega, \mathbb{C})$ or a monogenic function $v \in C^1(\Omega, \mathcal{A})$ gives directly the corresponding anti-holomorphic function $\widehat{v}$ because in $\mathbb{C}$ and $\mathcal{A}$ one have $\overline{v} = \widehat{v}$. For $\mathbb{H}$-valued monogenic functions this property doesn't hold in general as Proposition 1 shows.

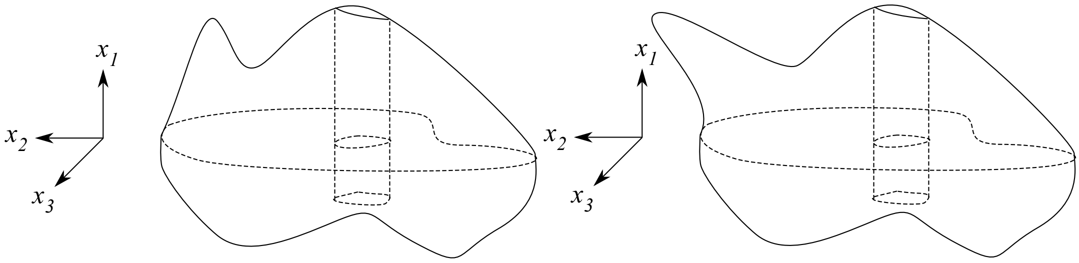
The geometrical restriction that apply to the domain in this paper is defined below.

Definition 4 (Domain normal with respect to the x_1-direction)**.** Let Ω be an open subset of $\mathbb{R}^3$, Ω is said normal with respect to the x_1-direction if there exists x_1^* such as for all $(x_1, x_2, x_3) \in \Omega$ and for all $x_1' \in \left[x_1^*, x_1\right]$ the point (x_1', x_2, x_3) is in Ω.

Basically domains normal with respect to the x_1-direction are constructed in two steps. First, a plane domain $\Omega^\perp \subset \text{span}(i, j)$ is defined without geometrical restriction. Then two real functions $\alpha(x_2, x_3)$ and $\beta(x_2, x_3)$ mapping from $\Omega^\perp$ to $\mathbb{R}$ define the upper and lower boundaries and:

$$\Omega = \left\{(x_1, x_2, x_3) \text{ such as } (x_2, x_3) \in \Omega^\perp \text{ and } x_1 = t\alpha(x_2, x_3) + (1-t)\beta(x_2, x_3), \forall t \in [0, 1]\right\}$$

Examples are presented in Figure 1.



(a) Domain normal with respect to the x_1-direction (b) Domain not normal with respect to the x_1-direction

Figure 1: Geometrical restrictions

The monogenic representation demonstrated in this paper relies on the following result, which has been demonstrated in a more general framework by Klein Obbink (1993) and more recently in the thesis of Álvarez-Peña (2013) or shortened in Álvarez-Peña and Porter (2014). A simple proof is reproduced here for sake of clarity.

Theorem 1 (Decomposition of harmonic functions into monogenic and anti monogenic functions)**.** Let Ω be on open subset of $\mathbb{R}^3$ normal with respect to the x_1-direction and let $f = f_1 + if_2 + jf_3$ be an harmonic function on $\Omega(\Delta f = 0)$. There exists a monogenic function Φ orthogonal to the set of monogenic constants and an anti-monogenic function Θ (more precisely $\Phi \in \ker \overline{\partial} \perp (\ker \partial \cap \ker \overline{\partial})$ and $\Theta \in \ker \partial$) such that:

$$f = \Phi + \Theta \tag{18}$$

Proof. Since the domain Ω is normal with respect to the x_1-direction there exists x_1^* such as one can define:

$$f^*(x_1, x_2, x_3) = \int_{x_1^*}^{x_1} f(t, x_2, x_3)\mathrm{d}t \tag{19}$$

It is easily verified that f^* is harmonic ($\Delta f^* = 0$) indeed:

$$\begin{aligned}
\Delta f^* &= \frac{\partial^2}{\partial x_1^2}\int_{x_1^*}^{x_1} f(t, x_2, x_3)\mathrm{d}t + \int_{x_1^*}^{x_1}\left(\frac{\partial^2}{\partial x_2^2} f(t, x_2, x_3) + \frac{\partial^2}{\partial x_3^2} f(t, x_2, x_3)\right)\mathrm{d}t \\
&= \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) - \frac{\partial}{\partial x_1} f(x_1^*, x_2, x_3) - \int_{x_1^*}^{x_1} \frac{\partial^2}{\partial x_1^2} f(t, x_2, x_3)\mathrm{d}t \\
&= \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) - \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = 0
\end{aligned} \tag{20}$$

Let introduce $\Phi = \frac{1}{2}\partial f^*$ and $\Theta = \frac{1}{2}\overline{\partial} f^*$. Since $\Delta = \partial\overline{\partial} = \overline{\partial}\partial$, $\overline{\partial}\Phi = 0$ and $\partial\Theta = 0$ thus Φ and Θ are respectively monogenic and anti-monogenic. Moreover:

$$\Phi + \Theta = \frac{1}{2}\left(\partial f^* + \overline{\partial} f^*\right) = \frac{\partial f^*}{\partial x_1} = f \tag{21}$$

This decomposition is not unique since for any monogenic constant $\Lambda \in (\ker\partial \cap \ker\overline{\partial})$ potentials $\Phi+\Lambda$ and $\Theta-\Lambda$ are still respectively monogenic and anti-monogenic. By setting Λ correctly one can consider that $\Phi \in \ker\overline{\partial} \perp (\ker\partial \cap \ker\overline{\partial})$. This will be constructively done in equation (42). □

4. Complete monogenic representation

4.1. Displacement field

In this section, a monogenic representation of displacement field is proposed with a proof of completeness using mathematical results of Section 3. The elastic domain is assumed to be normal with respect to the x_1-direction. The starting point is the Papkovich-Neuber complete representation reminded in Section 2. Let consider the $\mathbb{H}$-valued representation of the displacement vector $u = u_1 + iu_2 + ju_3$ the Papkovich-Neuber potential $f = f_1 + if_2 + jf_3$ and the potential related to the left side term of the Lamé-Navier equation $\Gamma^* = \Gamma_1^* + i\Gamma_2^* + j\Gamma_3^*$. Thus the bi-harmonic function (8) can be re-written:

$$G = \frac{1}{2}\left(\overline{x}f + \overline{f}x\right) + h, \tag{22}$$

Thus, the classical Papkovic-Neuber solution (11) reads in quaternionic algebra equivalently

$$2\mu u = 4(1-\nu)f - \left(\frac{\partial G}{\partial x_1} + i\frac{\partial G}{\partial x_2} + j\frac{\partial G}{\partial x_3}\right) + \Gamma^* = 4(1-\nu)f - \frac{1}{2}\overline{\partial}\left(\overline{x}f + \overline{f}x + 2h\right) + \Gamma^* \tag{23}$$

Now, since $f \in \ker\Delta$, Theorem 1 applies and there exist a decomposition of f, such that:

$$f = \Phi + \Theta \tag{24}$$

where $\Phi \in \ker\overline{\partial} \perp (\ker\partial \cap \ker\overline{\partial})$ defines a monogenic function orthogonal to the subset of monogenic constants, $\Theta \in \ker\partial$ an anti-monogenic function. This decomposition (24) is the explicit link between the presented monogenic representation and Papkovich-Neuber representation. Thus, applying the decomposition in (23) yields

$$\begin{aligned} 2\mu u = {} & 4(1-\nu)(\Phi+\Theta) - \tfrac{1}{2}\overline{\partial}\left(\overline{x}(\Phi+\Theta) + \overline{(\Phi+\Theta)}x + 2h\right) + \Gamma^* \\ = {} & 4(1-\nu)\Phi - \tfrac{1}{2}\overline{\partial}\left(\overline{x}\Phi + \overline{\Phi}x\right) - \tfrac{1}{2}\overline{\partial}\left(\overline{x}\Theta + \overline{\Theta}x\right) + 4(1-\nu)\Theta - \overline{\partial}h + \Gamma^* \end{aligned} \tag{25}$$

Now, it is easy to verify that:

(a) $\frac{1}{2}\overline{\partial}\left(\overline{x}\Theta + \overline{\Theta}x\right) = \overline{\partial}(x_1\Theta_1 + x_2\Theta_2 + x_3\Theta_3) \in \ker\partial$, since $\partial\Theta = 0$ one have:

$$\frac{1}{2}\partial\overline{\partial}\left(\overline{x}\Theta + \overline{\Theta}x\right) = x_1\Delta\Theta_1 + x_2\Delta\Theta_2 + x_3\Delta\Theta_3 + 2\left(\frac{\partial\Theta_1}{\partial x_1} + \frac{\partial\Theta_2}{\partial x_2} + \frac{\partial\Theta_3}{\partial x_3}\right) = 0. \tag{26}$$

(b) $\overline{\partial}h \in \ker\partial$,since $h \in \ker\Delta$.

(c) $4(1-\nu)\Theta \in \ker\partial$.

Therefore, there exist a monogenic function Ψ (cf Proposition 1) such as $\widehat{\Psi}$ is anti-monogenic and:

$$\widehat{\Psi} = \frac{1}{2}\overline{\partial}\left(\overline{x}\Theta + \overline{\Theta}x\right) - 4(1-\nu)\Theta + \overline{\partial}h \tag{27}$$

Hence from (25) the complete *generalized Kolosov-Muskhelishvili formula* for displacements reads as follows:

$$2\mu u = 4(1-\nu)\Phi - \frac{1}{2}\overline{\partial}\left(\overline{x}\Phi + \overline{\Phi}x\right) - \widehat{\Psi} + \Gamma^* \tag{28}$$

Or coordinate-wisely ($\Phi = \Phi_1 + i\Phi_2 + j\Phi_3 + k\Phi_4$ and $\widehat{\Psi} = \Psi_1 - i\Psi_2 - j\Psi_3 + \Psi_4$):

$$\begin{cases} 2\mu u_1 = 4(1-\nu)\Phi_1 - \dfrac{\partial}{\partial x_1}\left[x_1\Phi_1 + x_2\Phi_2 + x_3\Phi_3\right] - \Psi_1 + \Gamma_1^* \\ 2\mu u_2 = 4(1-\nu)\Phi_2 - \dfrac{\partial}{\partial x_2}\left[x_1\Phi_1 + x_2\Phi_2 + x_3\Phi_3\right] + \Psi_2 + \Gamma_2^* \\ 2\mu u_3 = 4(1-\nu)\Phi_3 - \dfrac{\partial}{\partial x_3}\left[x_1\Phi_1 + x_2\Phi_2 + x_3\Phi_3\right] + \Psi_3 + \Gamma_3^* \end{cases} \tag{29}$$

It should be noted that since $u = u_1 + iu_2 + ju_3 + ku_4$ with $u_4 = 0$ the the fourth component of (28) gives:

$$4(1-\nu)\Phi_4 - \Psi_4 = 0 \tag{30}$$

The latter condition (30) that arises in a natural way is essential for fixing linear dependencies when monogenic polynomials are used. It should be noted that any choice of monogenic functions Φ and Ψ satisfy the Lamé-Navier equations even if it does not fulfill (30), which generates an extra fourth component for the displacement but without interest. However in practice the best option is to seek monogenic potentials that fulfill (30). A further structural insight directly obtained from the extended hypercomplex formulation (28) is related to the representation of the bi-harmonic function G, which is by construction decomposed into a purely bi-harmonic part, i.e. $\mathrm{Sc}(\overline{x}\Phi)$ with $\Phi \in \ker\overline{\partial} \perp (\ker\partial \cap \ker\overline{\partial})$ and a purely harmonic part, i.e. $\mathrm{Sc}(\widehat{\Psi})$ with $\widehat{\Psi} \in \ker\partial$. The Papkovich-Neuber formulation does not allow such a direct decomposition.

The expression (28) is a complete (because the existence of potentials has been proven) representation of displacement field using only one monogenic function and one anti-monogenic function, thus 8 harmonic functions are needed, but it should be emphasized that monogenicity and anti-monogenicity (13) impose strong relationships between these 8 functions which lead to very interesting properties as pointed out in introduction. This is similar with Kolosov-Muskhelishvili formulas in 2D, two holomorphic functions are needed (which means 4 real-valued functions) although only one real bi-harmonic function is needed for the Airy potential, but holomorphy impose a strong relationship between the 4 real-valued functions, and interesting properties are obtained.

4.2. Stress field

Stress field is related to displacement field by the behavior in the equation set (1). Thus, by introducing $\underline{\boldsymbol{\epsilon}}^* = \underline{\boldsymbol{\epsilon}}^{th} + \underline{\boldsymbol{\epsilon}}^{res}$ and $\underline{\boldsymbol{\sigma}}^* = \underline{\boldsymbol{\sigma}} + \lambda\mathrm{tr}\left(\underline{\boldsymbol{\epsilon}}^*\right)\underline{\boldsymbol{I}} + 2\mu\underline{\boldsymbol{\epsilon}}^* = \lambda\mathrm{tr}\left(\underline{\boldsymbol{\epsilon}}\right)\underline{\boldsymbol{I}} + 2\mu\underline{\boldsymbol{\epsilon}}$ it is obtained:

$$\begin{cases} \sigma_{11}^* = \sigma_{11} + \left(\lambda\mathrm{tr}\left(\underline{\boldsymbol{\epsilon}}^*\right) + 2\mu\epsilon_{11}^*\right) = \lambda\left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3}\right) + 2\mu\dfrac{\partial u_1}{\partial x_1} \\ \sigma_{22}^* = \sigma_{22} + \left(\lambda\mathrm{tr}\left(\underline{\boldsymbol{\epsilon}}^*\right) + 2\mu\epsilon_{22}^*\right) = \lambda\left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3}\right) + 2\mu\dfrac{\partial u_2}{\partial x_2} \\ \sigma_{33}^* = \sigma_{33} + \left(\lambda\mathrm{tr}\left(\underline{\boldsymbol{\epsilon}}^*\right) + 2\mu\epsilon_{33}^*\right) = \lambda\left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3}\right) + 2\mu\dfrac{\partial u_3}{\partial x_3} \\ \sigma_{12}^* = \sigma_{12} + 2\mu\epsilon_{12}^* = \mu\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) \\ \sigma_{13}^* = \sigma_{13} + 2\mu\epsilon_{13}^* = \mu\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \sigma_{23}^* = \sigma_{23} + 2\mu\epsilon_{23}^* = \mu\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \end{cases} \tag{31}$$

The stress tensor $\underline{\boldsymbol{\sigma}}$ is obtained if $\underline{\boldsymbol{\sigma}}^*$ can be evaluated because $\underline{\boldsymbol{\epsilon}}^*$ is known. Thus $\operatorname{tr}\left(\underline{\boldsymbol{\sigma}}^*\right)$ is written:

$$\sigma_{11}^* + \sigma_{22}^* + \sigma_{33}^* = (3\lambda + 2\mu)\mathrm{Sc}\,[\partial u] \tag{32}$$

Let introduce following quantities related to displacements:

$$\widetilde{\sigma}_{12}^* = \mu\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \;\Bigg|\; \widetilde{\sigma}_{13}^* = \mu\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) \;\Bigg|\; \widetilde{\sigma}_{23}^* = \mu\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \tag{33}$$

Hence:

$$\begin{aligned}
-\sigma_{11}^* + \sigma_{22}^* + \sigma_{33}^* + 2i\sigma_{12}^* + 2j\sigma_{13}^* + 2k\widetilde{\sigma}_{23}^* &= \lambda\mathrm{Sc}\,[\partial u] - 2\mu\partial\widehat{u} \\
\sigma_{11}^* - \sigma_{22}^* + \sigma_{33}^* - 2i\sigma_{12}^* + 2j\widetilde{\sigma}_{13}^* + 2k\sigma_{23}^* &= \lambda\mathrm{Sc}\,[\partial u] - 2\mu i\partial(i\,\widehat{u}) \\
\sigma_{11}^* + \sigma_{22}^* - \sigma_{33}^* + 2i\widetilde{\sigma}_{12}^* - 2j\sigma_{13}^* - 2k\sigma_{23}^* &= \lambda\mathrm{Sc}\,[\partial u] - 2\mu j\partial(j\,\widehat{u})
\end{aligned} \tag{34}$$

A straightforward calculation gives complete *generalized Kolosov-Muskhelishvili formulas*:

$$\boxed{\begin{aligned}
2\mu u =&\; 4(1-\nu)\Phi - \frac{1}{2}\overline{\partial}\left(\overline{x}\Phi + \overline{\Phi}x\right) - \widehat{\Psi} + \Gamma^* \\
\sigma_{11}^* + \sigma_{22}^* + \sigma_{33}^* =&\; \frac{1+\nu}{1-2\nu}\mathrm{Sc}\,[2(1-2\nu)\partial\Phi + \partial\Gamma^*] \\
-\sigma_{11}^* + \sigma_{22}^* + \sigma_{33}^* + 2i\sigma_{12}^* + 2j\sigma_{13}^* + 2k\widetilde{\sigma}_{23}^* =&\; \frac{\nu}{1-2\nu}\mathrm{Sc}\,[2(1-2\nu)\partial\Phi + \partial\Gamma^*] + \partial\partial\left(\mathrm{Sc}\,[\overline{x}\Phi]\right) + \Psi - \partial\widehat{\Gamma^*} \\
\sigma_{11}^* - \sigma_{22}^* + \sigma_{33}^* - 2i\sigma_{12}^* + 2j\widetilde{\sigma}_{13}^* + 2k\sigma_{23}^* =&\; \frac{\nu}{1-2\nu}\mathrm{Sc}\,[2(1-2\nu)\partial\Phi + \partial\Gamma^*] \\
&-4(1-\nu)i\partial\left(i\widehat{\Phi}\right) + i\partial\left(i\partial\left(\mathrm{Sc}\,[\overline{x}\Phi]\right)\right) + i\partial\left(i\Psi\right) - i\partial\left(i\widehat{\Gamma^*}\right) \\
\sigma_{11}^* + \sigma_{22}^* - \sigma_{33}^* + 2i\widetilde{\sigma}_{12}^* - 2j\sigma_{13}^* - 2k\sigma_{23}^* =&\; \frac{\nu}{1-2\nu}\mathrm{Sc}\,[2(1-2\nu)\partial\Phi + \partial\Gamma^*] \\
&-4(1-\nu)j\partial\left(j\widehat{\Phi}\right) + j\partial\left(j\partial\left(\mathrm{Sc}\,[\overline{x}\Phi]\right)\right) + j\partial\left(j\Psi\right) - j\partial\left(j\widehat{\Gamma^*}\right)
\end{aligned}} \tag{35}$$

It can be noted that quantities $\widetilde{\sigma}_{12}^*$, $\widetilde{\sigma}_{13}^*$ and $\widetilde{\sigma}_{23}^*$ which are not of particular interest for an elastic problem, do not overlap the other components of the stress tensor. Thus these formal quantities simplifying the expression of stresses, do not disturb the classical boundary values problem.

The demonstrated monogenic representation has been developed as a refinement of classical harmonic Papkovich-Neuber representation because of the monogenicity of both potentials Φ and Ψ. Any choice of monogenic potentials leads to an elastic problem with some boundary conditions, equilibrium, behavior and compatibility are automatically verified. Monogenic functions constitute a subspace of harmonic functions, and therefore this paper enables to reduce the space where potentials are sought. Properties of monogenic functions are studied intensively. Expansion into power series (see e.q. Malonek (1990); Bock and Gürlebeck (2010); Bock (2012b)) and Laurent series (see e.g. van Lancker (1999); Bock (2012a)) are available. Conformal mapping technics are more limited than in 2D, but Möbius transformations of the form $(ax + b)(cx + d)^{-1}$ are available as demonstrated by Sudbery (1979).

5. Restriction to two-dimensions

This section aims at proving that the representation with two monogenic potentials presented in this paper is a straightforward generalization of the classical plane holomorphic representation developed by Muskhelishvili (1953b). Let begin with plane strain formulas. In this case potentials do not depend on x_3, moreover let consider that Φ and Ψ are two $\mathbb{C}$-valued functions (i.e. $\Phi_3 = \Phi_4 = \Psi_3 = \Psi_4 = 0$). Thus Φ and Ψ are holomorphic (because monogenicity coincides with holomorphy in 2D). Therefore commutativity is reestablished. Furthermore $z = x_1 + ix_2$, $\partial\Phi = 2\partial/\partial z - j\partial/\partial x_3$ and $\overline{\partial} = 2\partial/\partial\overline{z} + j\partial/\partial x_3$. Thus:

$$\begin{aligned} 2\mu u &= 4(1-\nu)\Phi - 2\frac{\partial}{\partial\overline{z}}\left(\frac{\overline{z}\Phi + z\overline{\Phi}}{2}\right) - \overline{\Psi} \\ &= (3-4\nu)\Phi - z\overline{\Phi'} - \overline{\Psi} \end{aligned} \tag{36}$$

For the stress field (35) gives:

$$\begin{aligned} &\sigma_{11} + \sigma_{22} + \sigma_{33} = 2(1+\nu)(\Phi' + \overline{\Phi'}) \\ &-\sigma_{11} + \sigma_{22} + \sigma_{33} + 2i\sigma_{12} + 2j\sigma_{13} = 2\nu(\Phi' + \overline{\Phi'}) + 2\overline{z}\Phi'' + 2\Psi' \\ &\sigma_{11} - \sigma_{22} + \sigma_{33} - 2i\sigma_{12} + 2k\sigma_{23} = 2\nu(\Phi' + \overline{\Phi'}) - 2\overline{z}\Phi'' - 2\Psi' \end{aligned} \tag{37}$$

Thus, from (36) and (37):

$$\begin{cases} \sigma_{33} = 2\nu(\Phi' + \overline{\Phi'}) \\ \sigma_{13} = \sigma_{23} = 0 \\ u_3 = 0 \end{cases} \tag{38}$$

Therefore the classic Muskhelishvili formulas for plane strain are obtained:

$$\begin{cases} 2\mu(u_1 + iu_2) = (3-4\nu)\Phi - z\overline{\Phi'} - \overline{\Psi} \\ \sigma_{11} + \sigma_{22} = 2(\Phi' + \overline{\Phi'}) \\ -\sigma_{11} + \sigma_{22} + 2i\sigma_{12} = 2(\overline{z}\Phi'' + \Psi') \end{cases} \tag{39}$$

Plane stress formulas are obtained by considering that $3-4\nu = (\lambda+3\mu)/(\lambda+\mu)$. Classically, plane stress problems verify the same equation set as in plane strain by replacing λ by $\lambda^* = 2\mu\lambda/(\lambda+2\mu)$. Thus $(\lambda^*+3\mu)/(\lambda^*+\mu) = (3-\nu)/(1+\nu)$ and the plane stress Muskhelishvili formulas are therefore obtained.

6. Orthogonal basis of solid spherical monogenics

Completeness of the generalized Kolosov-Muskhelishvili formulas has been proved. This section deals with the construction of a polynomial basis of Lamé solutions by using in particular the hypercomplex structure of the representation formulas. The corresponding problem of finding linear dependencies is well known as the uniqueness problem of the Papkovich-Neuber solutions (see e.g. Cong and Steven (1979b); Cong (1995) and references therein). Here are presented explicit conditions for fixing the linear dependencies which naturally arise from the properties and the finer structure of the function spaces used. To this end, the full quaternionic setting (28) is used which preserves all the structural properties of the functions.

Let us consider an orthogonal basis of monogenic polynomials with respect to the unit ball $\mathbb{B}_3$ in $\mathbb{R}^3$. This polynomial basis can be seen as a generalization of the holomorphic z-powers to $\mathbb{R}^3$ having special properties regarding the hypercomplex derivation and primitivation. In this section the basis elements are introduced by a two-step recurrence relation and some essential properties are highlighted. For a detailed explanation we refer to Bock and Gürlebeck (2010); Bock (2012b).

Proposition 2 (Bock (2012b))**.** For each $n \in \mathbb{N}$ and $l = 0, \ldots, n$, A_n^l denotes monogenic polynomials of degree n, that form an orthogonal basis of monogenic polynomials in $L^2(\Omega, \mathbb{H})$ satisfying the two-step recurrence formula:

$$A_{n+1}^l = \frac{n+1}{2(n-l+1)(n+l+2)} \left[\left((2n+3)x + (2n+1)\overline{x}\right)A_n^l - 2nx\overline{x}A_{n-1}^l\right] \tag{40}$$

with

$$A_{l+1}^l = \frac{1}{4}[(2l+3)x + (2l+1)\overline{x}]A_l^l \quad \text{and} \quad A_l^l = (x_1 - kx_2)^l \tag{41}$$

Note, that the initial values of the recurrence relation are defined by the subset of monogenic constants $\left\{A_l^l\right\}_{l\geq 0}$ which are polynomials isomorphic to the complex z-powers. We remark that the function Λ from Theorem 1 can be represented by the subset of monogenic constants $\left\{A_l^l\right\}_{l\geq 0}$. It is well known, e.g. Bauch (1981), that for each $n \in \mathbb{N}^*$ the polynomial solutions to the Lamé-Navier system of exact degree n form a subspace of dimension $6n+3$. Now, using (14) and (40) monogenic potentialΦ and anti-monogenic potential $\widehat{\Psi}$ are sought in form of polynomials expansion:

$$\Phi(x) = \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} A_n^l \alpha_{n,l} \quad \text{and} \quad \widehat{\Psi}(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \widehat{A}_m^k \beta_{m,k} \tag{42}$$

with $\alpha_{n,l}, \beta_{m,k} \in \mathbb{H}$. Let mention that monogenic constants are not considered in the polynomial expansion of $\Phi \in \ker \overline{\partial} \perp (\ker \partial \cap \ker \overline{\partial})$. Furthermore, it should be noted that polynomial basis of anti-monogenic functions was constructed by applying Proposition1 to the monogenic basis. Consequently, by substitution of the polynomial expansions in equation (28), we obtain with respect to the $\mathbb{R}$-linear space $4(2n+1) = 8n+4$ $\mathbb{H}$-valued polynomial solutions to the Lamé-Navier equation. The redundant polynomials of dimension $2n+1$ are fixed with the condition (30). This corresponds naturally to the dimension of the harmonic subspace, since by construction $4(1-\nu)\Phi_4 - \Psi_4 \in \ker \Delta$. For the polynomial basis (40) used here one could prove the following explicit $2n+1$ algebraic conditions:

Proposition 3. For each $n \in \mathbb{N}^*$ and using the polynomial expansions (42) in terms of the polynomial basis (40) in the extended displacement field (28), the $2n+1$ algebraic conditions such that $4(1-\nu)\Phi_4 - \Psi_4 = 0$ are given by:

$$\begin{aligned} 2\beta_{n,m+1}^1 - \beta_{n,m}^2 &= 4(1-\nu)[\alpha_{n,m}^2 + 2\alpha_{n,m+1}^1] \\ 2\beta_{n,m+1}^4 - \beta_{n,m}^3 &= 4(1-\nu)[\alpha_{n,m}^3 + 2\alpha_{n,m+1}^4] \\ \beta_{n,0}^4 &= 4(1-\nu)\alpha_{n,0}^4 \end{aligned} \tag{43}$$

with $m = 0, \ldots, n-1$. Note that for a compact representation the conventions $\alpha_{n,n}^1 = \alpha_{n,n}^4 = 0$ are used.

These conditions ensure that we obtain $6n+3$ $\mathcal{A}$-valued solutions to the Lamé-Navier equation equation and can be either included in the polynomial expansions or added as additional equations in the solution of the boundary value problem. Finally, some examples of the described scheme for the polynomial degrees $n = 0, 1, 2$ are given in Table 2. Symbolic mathematical programs such Mathematica or Maple can be used efficiently to generate automatically these independent polynomials. The corresponding displacements are obtained by replacing Φ and $\widehat{\Psi}$ coordinate-wisely in (28) by the ansatz functions of Table 2 and using (42).

Table 2: Ansatz functions and algebraic conditions for the extended displacement field

n	ansatz functions	coefficients	algebraic conditions
0	$\widehat{A_0^0} = 1$	$\beta_{0,0} \in \mathbb{H}$	$\beta_{0,0}^4 = 0$
1	$A_1^0 = x_1 + \frac{1}{2}(x_2 i + x_3 j)$	$\alpha_{1,0} \in \mathbb{H}$	$2\beta_{1,1}^1 - \beta_{1,0}^2 = 4(1-\nu)\alpha_{1,0}^2$
	$\widehat{A_1^0} = x_1 - \frac{1}{2}(x_2 i + x_3 j)$	$\beta_{1,0}, \beta_{1,1} \in \mathbb{H}$	$2\beta_{1,1}^4 - \beta_{1,0}^3 = 4(1-\nu)\alpha_{1,0}^3$
	$\widehat{A_1^1} = x_2 - x_3 k$		$\beta_{1,0}^4 = 4(1-\nu)\alpha_{1,0}^4$
2	$A_2^0 = x_1^2 - \frac{1}{2}(x_2^2 + x_3^2) + x_1x_2 i + x_1x_3 j$	$\alpha_{2,0}, \alpha_{2,1} \in \mathbb{H}$	$2\beta_{2,1}^1 - \beta_{2,0}^2 = 4(1-\nu)[\alpha_{2,0}^2 + 2\alpha_{2,1}^1]$
	$A_2^1 = 2x_1x_2 + \frac{1}{2}(x_2^2 - x_3^2)i + x_2x_3 j - 2x_1x_3 k$	$\beta_{2,0}, \beta_{2,1}, \beta_{2,2} \in \mathbb{H}$	$2\beta_{2,2}^1 - \beta_{2,1}^2 = 4(1-\nu)\alpha_{2,1}^2$
	$\widehat{A_2^0} = x_1^2 - \frac{1}{2}(x_2^2 + x_3^2) - x_1x_2 i - x_1x_3 j$		$2\beta_{2,1}^4 - \beta_{2,0}^3 = 4(1-\nu)[\alpha_{2,0}^3 + 2\alpha_{2,1}^4]$
	$\widehat{A_2^1} = 2x_1x_2 - \frac{1}{2}(x_2^2 - x_3^2)i - x_2x_3 j - 2x_1x_3 k$		$2\beta_{2,2}^4 - \beta_{2,1}^3 = 4(1-\nu)\alpha_{2,1}^3$
	$\widehat{A_2^2} = x_2^2 - x_3^2 - 2x_2x_3 k$		$\beta_{2,0}^4 = 4(1-\nu)\alpha_{2,0}^4$

7. Conclusion and outlook

One of most fruitful and elegant method for elastic plane problems has been established by Muskhelishvili (1953b) by using only two complex-valued holomorphic potentials. In this paper, an extension in 3D has been demonstrated by using two quaternionic-valued monogenic potentials, which appears to be a suitable extension in higher dimensions of classical holomorphic functions. The obtained monogenic representation is compact and a straightforward calculation shows that classical Muskhelishivli formulas in 2D is embedded in the extended formulation. Completeness is demonstrated with classical tools of potential theory. Geometrical restrictions have been specified. This leads to a very wide class of possible shapes for the elastic body, and more general shapes can be considered by solving the elastic problem on subparts that verify the geometrical restrictions.

The obtained monogenic formulation of the three dimensional elasticity problem represents a refinement of the classical harmonic Papkovich-Neuber solution. Due to the factorization of the 2nd order Laplace operator by the 1st order generalized Cauchy-Riemann operator and its adjoint operator, two vector-valued monogenic functions have to be find (i.e., eight harmonic functions related to each other by a strong relationship) instead of four real-valued harmonic functions. This is similar to the situation in 2D. A significant advantage of such a hypercomplex representation is when approximate solutions of boundary value problems are sought using series expansions of homogeneous polynomials. In Bock (2009) it was shown that the properties (e.g. orthogonality, Appell property, orthogonal decomposition into higher and lower dimensional subspaces) of the polynomial systems used to approximate the monogenic potentials Φ and Ψ improve immediately the numerical properties of the resulting polynomial solutions to the Lamé-Navier equation even if these polynomial solutions no longer have the mentioned properties. Moreover for the significant issue of finding polynomial approximations, structural properties of monogenic basis (e.g. Bock and Gürlebeck (2009a)) enables to fix explicitly linear dependencies generated by polynomial potentials. In a more general context this is known as the uniqueness problem of the Papkovich-Neuber solution (see e.g. Cong and Steven (1979b)). There it is proved that under certain geometric restrictions (star-shaped or domains normal with respect to x_1-direction) one of the harmonic potentials can be neglected from the representation formula. For general simply connected domains this is not valid.

This contribution differs from existing related works by using an approach not relying on polynomial subspaces but a constructive method that proves the existence of the monogenic potentials and thus completeness of the representation. These efforts (previous and present works) help to understand better the structure behind the representation uniqueness and possibly overcome the difficulty.

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